

Hitting Sets Online and Unique-Max Coloring

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July 12, 2012

Abstract

We consider the problem of hitting sets online. The hypergraph (i.e., range-space consisting of points and ranges) is known in advance, and the ranges to be stabbed are input one-by-one in an online fashion. The online algorithm must stab each range upon arrival. An online algorithm may add points to the hitting set but may not remove already chosen points. The goal is to use the smallest number of points. The best known competitive ratio for hitting sets online by Alon et al. [AAA⁺09] is $O(\log n \cdot \log m)$ for general hypergraphs, where n and m denote the number of points and the number of ranges, respectively.

We consider hypergraphs in which the union of two intersecting ranges is also a range. Our main result for such hypergraphs is as follows. The competitive ratio of the online hitting set problem is at most the unique-max number and at least this number minus one.

Given a graph $G = (V, E)$, let $H = (V, R)$ denote the hypergraph whose hyperedges are subsets $U \subseteq V$ such that the induced subgraph $G[U]$ is connected. We establish a new connection between the best competitive ratio for the online hitting set problem in H and the vertex ranking number of G . This connection states that these two parameters are equal. Moreover, this equivalence is algorithmic in the sense, that given an algorithm to compute a vertex ranking of G with k colors, one can use this algorithm as a black-box in order to design a k -competitive deterministic online hitting set algorithm for H . Also, given a deterministic k -competitive

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online algorithm for H , we can use it as a black box in order to compute a vertex ranking for G with at most k colors. As a corollary, we obtain optimal online hitting set algorithms for many such hypergraphs including those realized by planar graphs, graphs with bounded tree width, trees, etc. This improves the best previously known general bound of Alon et al. [AAA⁺09].

We also consider two geometrically defined hypergraphs. The first one is defined by subsets of a given set of n points in the Euclidean plane that are induced by half-planes. We obtain an $O(\log n)$ -competitive ratio. We also prove an $\Omega(\log n)$ lower bound for the competitive ratio in this setting. The second hypergraph is defined by subsets of a given set of n points in the plane induced by unit discs. Since the number of subsets in this setting is $O(n^2)$, the competitive ratio obtained by Alon et al. is $O(\log^2 n)$. We introduce an algorithm with $O(\log n)$ -competitive ratio. We also show that any online algorithm for this problem has a competitive ratio of $\Omega(\log n)$, and hence our algorithm is optimal.

1 Introduction

In the minimum hitting set problem, we are given a hypergraph (X, R) , where X is the ground set of points and R is a set of hyperedges. The goal is to find a “small” cardinality subset $S \subseteq X$ such that every hyperedge is stabbed by S , namely, every hyperedge has a nonempty intersection with S .

The minimum hitting set problem is a classical NP-hard problem [Kar72], and remains hard even for geometrically induced hypergraphs (see [HM85] for several references). A sharp logarithmic threshold for hardness of approximation was proved by Feige [Fei98]. On the other hand, the greedy algorithm achieves a logarithmic approximation ratio [Joh74, Chv79]. Better approximation ratios have been obtained for several geometrically induced hypergraphs using specific properties of the induced hypergraphs [HM85, KR99, BMKM05]. Other improved approximation ratios are obtained using the theory of VC-dimension and ε -nets [BG95, ERS05, CV07]. Much less is known about online versions of the hitting set problem.

In this paper, we consider an online setting in which the set of points X is given in the beginning, and the ranges are introduced one by one. Upon arrival of a new range, the online algorithm may add points (from X) to the hitting set so that the hitting set

also stabs the new range. However, the online algorithm may not remove points from the hitting set. We use the competitive ratio for our analysis, a classical measure for the performance of online algorithms [ST85, BEY98].

Alon et al. [AAA⁺09] considered the online set-cover problem for arbitrary hypergraphs. In their setting, the ranges are known in advance, and the points are introduced one by one. Upon arrival of an uncovered point, the online algorithm must choose a range that covers the point. Hence, by interchanging the roles of ranges and points, the online set-cover problem and the online hitting-set problems are equivalent. The online set cover algorithm presented by Alon et al. [AAA⁺09] achieves a competitive ratio of $O(\log n \log m)$ where n and m are the number of points and the number of hyperedges respectively. Note that if $m \geq 2^{n/\log n}$, the analysis of the online algorithm only guarantees that the competitive ratio is $O(n)$; a trivial bound if one range is chosen for each point.

Unique-maximum coloring. We consider two types of colorings. A coloring $c : X \rightarrow [0, k]$ is a *unique-max coloring* of a hypergraph $H = (X, R)$ if, for each range $r \in R$, exactly one point is colored by the color $\max_{x \in r} c(x)$ (c.f., [Smo12]). A *vertex ranking* (also an *ordered coloring*) of a graph $G = (V, E)$ is a coloring of the vertices $c : V \rightarrow \{1, \dots, k\}$ that satisfies the following property. Every simple path, the endpoints of which have the same color i , contains a vertex with a color greater than i [KMS95, Sch89].

Relation between unique-maximum coloring and the competitive ratio. We consider the competitive ratio for the hitting set problem as a property of a hypergraph. Namely, the competitive ratio of a hypergraph $H = (X, R)$ is the competitive ratio of the best deterministic online algorithm for the hitting set problem over H . We say that a hypergraph is *I*-type if the union of two intersecting ranges is always a range. Our main result (Theorem 7) shows a new connection between the competitive ratio of an *I*-type hypergraph H and the minimum number of colors required to color H in a unique-max coloring. In fact, we present “black box” reductions that construct an online hitting set algorithm from a unique-max coloring, and vice-versa.

Applications. Three applications of the main result are presented. The first application is motivated by the following setting. Consider a communication network $G = (V, E)$. This network is supposed to serve requests for virtual private networks (VPNs). Each VPN request is a subset of vertices that induces a connected subgraph in the network, and requests for VPNs arrive online. For each VPN, we need to assign a server (among the nodes in the VPN) that serves the nodes of the VPN. Since setting up a server is expensive, the goal is to select as few servers as possible.

This application can be abstracted by considering hypergraphs H that are realized as the connected induced subgraphs of a given graph G . This hypergraph captures the online problem in which the adversary chooses subsets $V' \subseteq V$ such that the induced subgraph $G[V']$ is connected, and the algorithm must stab the subgraphs. A direct consequence of the observation that every unique-max coloring of H is a vertex ranking of G implies that the competitive ratio of H equals the vertex ranking number of G . This application leads to improved optimal competitive ratios for graphs that admit (hereditary) small balanced separators (see Table 1).

Two more classes of hypergraphs are obtained geometrically as follows. In both settings we are given a set X of n points in the plane. In one hypergraph, the hyperedges are intersections of X with half planes. In the other hypergraph, the hyperedges are intersections of X with unit discs. Although these hypergraphs are not I -type, we present an online algorithm for the hitting set problem for points in the plane and unit discs (or half-planes) with an optimal competitive ratio of $O(\log n)$. The competitive ratio of this algorithm improves the $O(\log^2 n)$ -competitive ratio of Alon et al. by a logarithmic factor.

An application for points and unit discs is the selection of access points or base stations in a wireless network. The points model base stations and the disc centers model clients. The reception range of each client is a disc, and the algorithm has to select a base station that serves a new uncovered client. The goal is to select as few base stations as possible.

Organization. Definitions and notation are presented in Section 2. In Section 3, we study the special case of intervals on a line. The main result is presented in Section 4. We apply the main result to hypergraphs induced by connected subgraphs of a given graph in Section 5. An online algorithm for hypergraphs induced by points and half-

planes is presented in Section 6. An online algorithm for the case of points and unit discs is presented in Section 7. We conclude with open problems.

2 Preliminaries

The online minimum hitting set problem. Let $H = (X, R)$ denote a hypergraph, where R is a set of nonempty subsets of the ground set X . Members in X are referred to as *points*, and members in R are referred to as *ranges* (or *hyperedges*). A subset $S \subseteq X$ *stabs* a range r if $S \cap r \neq \emptyset$. A *hitting set* is a subset $S \subseteq X$ that stabs every range in R . In the minimum hitting set problem, the goal is to find a hitting set with the smallest cardinality.

In this paper, we consider the following online setting. The adversary introduces a sequence $\sigma \triangleq \{r_i\}_{i=1}^s$ of ranges. Let σ_i denote the prefix $\{r_1, \dots, r_i\}$. The online algorithm must compute a chain of hitting sets $C_1 \subseteq C_2 \subseteq \dots$ such that C_i is a hitting set with respect to the ranges in σ_i . In other words, upon arrival of the range r_i , if r_i is not stabbed by C_{i-1} , then the online algorithm adds a point $x_i \in r_i$ to C_{i-1} so that $C_i \triangleq C_{i-1} \cup \{x_i\}$ stabs all the ranges in σ_i . If C_{i-1} stabs the range r_i , then the algorithm need not add a point, and $C_i \triangleq C_{i-1}$.

Fix a hypergraph H and an online deterministic algorithm ALG . The competitive ratio of the algorithm ALG with respect to H is defined as follows. For a finite input sequence $\sigma = \{r_i\}_{i=1}^s$, let $\text{OPT}(\sigma_i) \subseteq X$ denote a minimum cardinality hitting set for the ranges in σ_i . Let $\text{ALG}(\sigma) \subseteq X$ denote the hitting set computed by an online algorithm ALG when the input sequence is σ . Note that the sequence of minimum hitting sets $\{\text{OPT}(\sigma_i)\}_{i=1}^s$ is not necessarily a chain of inclusions.

Definition 1. *The competitive ratio of a deterministic online hitting set algorithm ALG is defined by*

$$\rho_H(\text{ALG}) \triangleq \max_{\sigma} \frac{|\text{ALG}(\sigma)|}{|\text{OPT}(\sigma)|}.$$

The competitive ratio of the hypergraph H is defined by

$$\rho_H \triangleq \min_{\text{ALG}} \rho_H(\text{ALG}).$$

The definition of ρ_H can be viewed as a hypergraph property. It equals the best competitive ratio achievable by *any* online deterministic algorithm with respect to the hypergraph H .

I-type hypergraphs. We now define a notion that captures an important property of the hypergraph of intervals over collinear points.

Definition 2. A hypergraph $H = (X, R)$ is I-type if it satisfies the following property:

$$\forall r_1, r_2 \in R : r_1 \cap r_2 \neq \emptyset \Rightarrow r_1 \cup r_2 \in R.$$

Unique-max colorings. Consider a hypergraph $H = (X, R)$ and a coloring $c : X \rightarrow \mathbb{N}$. For a range $r \in R$, let $c_{\max}(r) \triangleq \max\{c(x) \mid x \in r\}$. Similarly, $c_{\min}(r) \triangleq \min\{c(x) \mid x \in r\}$.

Definition 3. A coloring $c : X \rightarrow \mathbb{N}$ is a unique-max coloring of $H = (X, R)$ if, for every range $r \in R$, there is a unique point $x \in r$ for which $c(x) = c_{\max}(r)$.

Similarly, a coloring is unique-min if, for every range r , exactly one point $x \in r$ is colored $c_{\min}(r)$.

The *unique-max-chromatic number* of a hypergraph H , denoted by $\chi_{um}(H)$, is the least integers k for which H admits a unique-maximum coloring that uses only k colors.

Vertex ranking. We define a coloring notion for graphs known as vertex ranking [KMS95, Sch89].

Definition 4. A vertex ranking of a graph $G = (V, E)$ is a coloring $c : V \rightarrow \mathbb{N}$ that satisfies the following property. For every pair of distinct vertices x and y and for every simple path P from x to y , if $c(x) = c(y)$, then there exists an internal vertex z in P such that $c(x) < c(z)$.

The *vertex ranking number* of G , denoted $\chi_{vr}(G)$, is the least integer k for which G admits a vertex ranking that uses only k colors.

A vertex ranking of a graph G is also a proper coloring of G since adjacent vertices must be colored by different colors. On the other hand, a proper coloring is not necessarily a vertex ranking as is easily seen by considering a path graph P_n . This graph

admits a proper coloring with 2 colors but this coloring is not a valid vertex ranking. In fact, $\chi_{\text{vr}}(P_n) = \lfloor \log_2 n \rfloor + 1$, as proved in the following proposition that has been proven several times [IRV88, KMS95, ELRS03].

Proposition 5. $\chi_{\text{vr}}(P_n) = \lfloor \log_2 n \rfloor + 1$.

Proof. Consider a vertex ranking c of P_n in which the highest color is used once to split the path into two disjoint paths as evenly as possible. The number of colors $f(n)$ satisfies the recurrence $f(1) = 1$ and

$$f(n) \leq 1 + f\left(\left\lceil \frac{n-1}{2} \right\rceil\right).$$

It is easy to verify that $f(n) \leq 1 + \lfloor \log_2 n \rfloor$. For a matching lower bound, consider a coloring and the point with the highest color. Note that the highest color appears uniquely in P_n . This point separates the path into two disjoint paths colored by one color less. The length of one of these paths must be at least $\lceil \frac{n-1}{2} \rceil$. Hence $f(n) \geq 1 + f(\lceil \frac{n-1}{2} \rceil)$ and therefore $f(n) \geq 1 + \lfloor \log_2 n \rfloor$, as required. \square

3 Special Case: Hitting set for Intervals on the Line

Consider the hypergraph $H = (X, R)$ of intervals over n collinear points defined by:

$$\begin{aligned} X &\triangleq \{1, 2, \dots, n\} \\ R &\triangleq \{[i, j] \mid 1 \leq i \leq j \leq n\}. \end{aligned}$$

The competitive ratio of the online hitting-set algorithm of Alon et al. [AAA⁺09] for the hypergraph of intervals over n collinear points is $O(\log |X| \cdot \log |R|) = O(\log^2 n)$. In this section we prove a better competitive ratio for this specific hypergraph.

Proposition 6. $\rho(H) = \lfloor \log_2 n \rfloor + 1$.

Proof. We begin by proving the lower bound $\rho(H) \geq \lfloor \log_2 n \rfloor + 1$. The adversary generates the sequence $\sigma \triangleq \{r_i\}$ of ranges to be stabbed. Let $\{C_i\}_i$ denote the chain of hitting sets computed by the algorithm. The first range consists of all the points, namely, $r_1 = X$. In every step, the next range r_{i+1} is chosen to be a larger interval in $r_i \setminus C_i$,

namely, $|r_{i+1}| \geq \frac{|r_i|-1}{2}$. While r_i is not empty, the adversary forces the algorithm to stab each range by a distinct point. In fact, the adversary can introduce such sequence consisting of at least $\lfloor \log_2 n \rfloor + 1$ many ranges. Thus, $|C_i| = i$ if $i \leq \lfloor \log_2 n \rfloor + 1$. However, $r_1 \supset r_2 \supset \dots$ is a decreasing chain, and hence, $|\text{OPT}(\sigma_i)| = 1$, and the lower bound follows.

The upper bound $\rho(H) \leq \lfloor \log_2 n \rfloor + 1$ is proved as follows. Let $c(x)$ denote a vertex ranking of the graph P_n that uses $\lfloor \log_2 n \rfloor + 1$ colors (see Prop. 5). Consider the deterministic hitting-set algorithm ALG_c defined as follows. Upon arrival of an unstabbed interval $[i, j]$, stab it by the point x in the interval $[i, j]$ with the highest color. Namely $x \triangleq \arg \max\{c(k) : i \leq k \leq j\}$.

We claim that $\rho_H(\text{ALG}_c) \leq 1 + \lfloor \log_2 n \rfloor$. The proof is based on the following observation. Consider a color γ and the subsequence of intervals $\sigma(\gamma)$ that consists of the intervals r_i in σ that satisfy the following two properties: (i) Upon arrival r_i is unstabbed. (ii) Upon arrival of r_i , ALG_c stabs r_i by a point colored γ . We claim that the intervals in $\sigma(\gamma)$ are pairwise disjoint. Indeed, if two intervals $r_1 \neq r_2$ in $\sigma(\gamma)$ intersect, then the maximum color in $r_1 \cup r_2$ is also γ , and it appears twice in $r_1 \cup r_2$. This contradicts the definition of a vertex ranking because $r_1 \cup r_2$ is also an interval. Thus, the optimum hitting set satisfies $|\text{OPT}(\sigma)| \geq \max_\gamma |\sigma(\gamma)|$. But $|\text{ALG}_c(\sigma)| \leq (1 + \lfloor \log_2 n \rfloor) \cdot \max_\gamma |\sigma(\gamma)|$, and hence $\rho_H(\text{ALG}_c) \leq 1 + \lfloor \log_2 n \rfloor$, as required. \square

4 The Main Result

Theorem 7. *If a hypergraph $H = (X, R)$ is I-type, then*

$$\chi_{um}(H) - 1 \leq \rho(H) \leq \chi_{um}(H).$$

The proof of Theorem 7 is by black-box reductions. The first reduction uses the unique-max coloring to obtain an online algorithm (simply stab a range with the point with the highest color). The second reduction uses a deterministic online hitting set algorithm to obtain a unique-max coloring.

We say that a hypergraph $H = (X, E)$ is *separable* if $\{x\} \in R$, for every $x \in X$. The proof of the following corollary appears in Section 4.3.

Corollary 8. *If a hypergraph $H = (X, R)$ is I -type and separable, then $\rho(H) = \chi_{um}(H)$.*

4.1 Proof of $\rho(H) \leq \chi_{um}(H)$

The proof follows the reduction in the proof of Prop. 6. Let $k = \chi_{um}(H)$ and let $c : X \rightarrow [1, k]$ denote a unique-max coloring of $H = (X, R)$. Consider the deterministic hitting-set algorithm ALG_c defined as follows. Upon arrival of an unstabbed range $r \in R$, stab it by the point $x \in r$ colored $c_{\max}(r)$.

We claim that $\rho_H(\text{ALG}_c) \leq k$. Fix a sequence $\sigma = \{r_i\}_i$ of ranges input by the adversary. For a color γ , let $\sigma(\gamma)$ denote the subsequence of σ that consists of the ranges r_i in σ that satisfy the following properties: (i) r_i is unstabbed when it arrives. (ii) The first point that ALG_c uses to stab r_i is colored γ . The ranges in $\sigma(\gamma)$ are pairwise disjoint. Indeed, if two ranges $r_1 \neq r_2$ in $\sigma(\gamma)$ intersect, then $r_1 \cup r_2 \in R$. Moreover, the maximum color in $r_1 \cup r_2$ is also γ . But the color γ appears twice in the range $r_1 \cup r_2$; one point that stabs r_1 and another point that stabs r_2 , a contradiction. Thus the optimum hitting set satisfies $\text{OPT}(\sigma) \geq \max_{\gamma} |\sigma(\gamma)|$. But

$$\text{ALG}_c(\sigma) = \sum_{\gamma=1}^k |\sigma(\gamma)| \leq k \cdot \max_{\gamma} |\sigma(\gamma)|.$$

and hence $\rho_H(\text{ALG}_c) \leq k$, as required.

4.2 Proof of $\chi_{um}(H) \leq 1 + \rho(H)$

Let ALG denote a deterministic online hitting set algorithm that satisfies $\rho_H(\text{ALG}) = \rho(H)$. We use ALG as a “black box” to compute a unique-min coloring $c : X \rightarrow [0, \rho(H)]$. Note that we compute a unique minimum coloring rather than a unique maximum coloring; this modification simplifies the presentation. (If $c(x)$ is a unique-min coloring, then $c'(x) \triangleq \rho(H) - c(x)$ is a unique-max coloring.)

Terminology. Let $S \subseteq X$ be a subset of points. We say that a range $r \in R$ is S -*maximal* if no range contained in S strictly contains r . Formally, for every range $r' \in R$, $r \subseteq r' \subseteq S$ implies that $r' = r$. Given a node v in a rooted tree, let $\text{path}(v)$ denote the

path from the root to v . Define $\text{depth}(v)$ to be the distance from the root to v . (The distance of the root to itself is zero.) The *least common ancestor* of two nodes u and v in a rooted tree is the node of highest depth in $\text{path}(u) \cap \text{path}(v)$.

4.2.1 The Decomposition

We use ALG to construct a decomposition forest consisting of rooted trees. Each node v in the forest is labeled by a range $r_v \in R$ and a point $x_v \in r_v$. The decomposition forest is defined inductively as follows.

For each X -maximal range in R we associate a distinct root. The labels of each root v are defined as follows. The range r_v is the X -maximal range associated with v . The point $x_v \in r_v$ is the point that ALG uses to stab r_v when the input sequence consists only of r_v .

We now describe the induction step for defining the children of a node v and its labels r_v and x_v . Let $X(\text{path}(v)) \triangleq \{x_u \mid u \in \text{path}(v)\}$ denote the sequence of points that appear along the path from the root to v . Similarly, let $\sigma(\text{path}(v))$ denote the sequence of ranges that appear along $\text{path}(v)$. Let $S \triangleq r_v \setminus X(\text{path}(v))$. For each nonempty S -maximal range r , we add a child v' of v that is labeled by the range $r_{v'} = r$. The point $x_{v'}$ is the point x that stabs $r_{v'}$ when ALG is input the sequence of ranges $\sigma(\text{path}(v'))$. We stop with a leaf v if there is no range contained in $S \setminus X(\text{path}(v))$.

Proposition 9. *For every node v , the sequence of ranges in $\sigma(\text{path}(v))$ is a strictly decreasing chain. Namely, if v is a child of u then $r_v \subsetneq r_u$. Moreover, when this sequence is input to ALG, then each range is unstabbed upon arrival. Hence, the points in $X(\text{path}(v))$ are distinct.*

Proposition 10. *If v_1 and v_2 are siblings, then the ranges r_{v_1} and r_{v_2} are disjoint.*

Proof. Otherwise, since H is I -type, $r_{v_1} \cup r_{v_2}$ is a range. This range contradicts the S -maximality of r_{v_1} and r_{v_2} for $S \triangleq r_v \setminus X(\text{path}(v))$, where v is the parent of v_1 and v_2 . \square

Proposition 11. *If v and u are two nodes such that v is neither an ancestor or a descendant of u , then the ranges r_v and r_u are disjoint.*

Proof. For the sake of contradiction, assume that $x \in r_u \cap r_v$. It follows that u and v must belong to the same tree whose root is labeled by the X -maximal range that contains x . The least common ancestor w of u and v has two distinct children w_1 and w_2 such that $w_1 \in \text{path}(u)$ and $w_2 \in \text{path}(v)$. By Proposition 9, $r_u \subset r_{w_1}$ and $r_v \subset r_{w_2}$. By Proposition 10, $r_{w_1} \cap r_{w_2} = \emptyset$, and it follows that $r_u \cap r_v = \emptyset$, as required. \square

The following proposition is an immediate consequence of Propositions 9 and 11.

Proposition 12. *All the labels x_v of the nodes in the forest are distinct.*

4.2.2 Mapping ranges to nodes in the decomposition

Let \tilde{X} denote the set of nodes in the decomposition forest. We now define a mapping $f : R \rightarrow \tilde{X}$ from the set of ranges R to the set of nodes of the decomposition forest.

Define $f(r)$ to be the forest node v of minimum depth such that x_v stabs r . Formally,

$$T(r) \triangleq \{v \in \tilde{X} \mid x_v \in r\}$$

$$f(r) \triangleq \arg \min\{\text{depth}(v) \mid v \in T(r)\}.$$

Claim 13. *The mapping $f(r)$ is well defined.*

Proof. We need to prove that (1) $T(r)$ is not empty for every range r , and (2) there exists a unique forest node $v \in T(r)$ of minimum depth.

We prove that $T(r) \neq \emptyset$ by contradiction. Let $x \in r$ be any point in r . Consider the X -maximal range r_1 that contains x . Let v_1 be the root that is associated with r_1 (i.e., $r_{v_1} = r_1$). Clearly $r \subset r_{v_1}$ because $x \in r$ and r_{v_1} is X -maximal. By the assumption, $x_{v_1} \notin r$. Proceed along the tree rooted at v_1 to find a tree path v_1, v_2, \dots, v_k such that $r \subset r_{v_i}$ and $x_{v_i} \notin r$ for $1 \leq i \leq k$. To obtain a contradiction, we claim that one can find such an infinite path since $r \subseteq r_{v_k} \setminus \{x_{v_k}\}$. Indeed, r is contained in one of the S -maximal ranges for $S \triangleq r_{v_k} \setminus X(\text{path}(v_k))$. So we can define v_{k+1} to be the child of v_k such that $r \subseteq r_{v_{k+1}}$. Since $T(r)$ is empty $x_{v_{k+1}} \notin r$, the node v_{k+1} meets the requirement from the next node in the path. However, by Proposition 9, each tree in the forest is finite, a contradiction.

We prove that there exists a unique forest node $v \in T(r)$ of minimum depth. Assume that there are two forest nodes u and v of minimum depth such that both u and v are in

$T(r)$. By the definition of $S(r)$, $x_u \in r$. By the fact that $\text{depth}(u)$ is minimum it follows that $r \cap X(\text{path}(u)) = \{x_u\}$. Hence, by the maximality of r_u , it follows that $r \subseteq r_u$. Analogously, $r \subseteq r_v$. Hence, r_u and r_v are not disjoint. By Proposition 11, u is an ancestor of v , or vice-versa. This implies that $\text{depth}(u) \neq \text{depth}(v)$, a contradiction. \square

4.2.3 The Coloring

Define the coloring $c : X \rightarrow \mathbb{N}$ as follows. For each $x \in X$, if $x = x_v$ for some forest node v , then define $c(x) \triangleq \text{depth}(v)$. If x does not appear as a label x_v of any node in the forest, then $c(x) \triangleq \rho_H(\text{ALG})$. Note that Proposition 12 insures that the coloring c is well defined.

Lemma 14. *The depth of every forest node is less than $\rho_H(\text{ALG})$.*

Proof. Consider a node v in the decomposition forest. By Proposition 9, the sequence $\sigma(\text{path}(v))$ of ranges is a decreasing chain, and when input to ALG, each range is unstabbed upon arrival. Therefore the cardinality of the hitting set that $\text{ALG}(\sigma(\text{path}(v)))$ returns equals $1 + \text{depth}(v)$. On the other hand, x_v stabs all these ranges. Since the competitive ratio of ALG with respect to H is $\rho_H(\text{ALG})$, it follows that the length of this sequence is not greater than $\rho_H(\text{ALG})$. The length of this sequence equals $1 + \text{depth}(v)$, and the lemma follows. \square

Lemma 14 implies that the maximum color assigned by $c(x)$ is $\rho_H(\text{ALG})$. The following lemma implies that $\chi_{um}(H) \leq \rho_H(\text{ALG}) + 1$.

Lemma 15. *The coloring $c : X \rightarrow [0, \rho_H(\text{ALG})]$ is a unique-min coloring.*

Proof. Fix a range r . By Claim 13, $f(r)$ is well defined. Thus r contains only one point colored $c(r)$, and all the other points in r are colored by higher colors. \square

We remark that the proof of Lemma 15 uses the color $\rho_H(\text{ALG})$ as a "neutral" color that is never used as the minimum color in a range.

4.3 Proof of Corollary 8

Lemma 16. *If H is separable, then every point appears as a label x_v in the decomposition.*

Proof. If H is separable, then the stopping condition in the construction of the decomposition trees is equivalent to $r_v = \{x_v\}$. Otherwise, for each point x in $r_v \setminus \{x_v\}$, the range $\{x\}$ excludes the possibility that v is a leaf. This implies that every point appears as a label of a node in the decomposition forest, as required. \square

Proof of Corollary 8. A point x is colored $\rho_H(\text{ALG})$ iff no node is labeled by x in the decomposition forest. By Lemma 16, if H is separable, then every point appears as a label x_v in the decomposition. Thus, the color $\rho_H(\text{ALG})$ is never used by $c(x)$. Hence the range of the coloring $c(x)$ is $[0, \rho_H(\text{ALG}) - 1]$ and the number of colors used by $c(x)$ is only $\rho_H(\text{ALG})$, as required. \square

5 Online Hitting-Set for Connected Subgraphs

We consider the following setting of a hypergraph induced by connected subgraphs of a given graph. Formally, let $G = (V, E)$ be a graph. Let $H = (V, R)$ denote the hypergraph over the same set of vertices V . A subset $r \subseteq V$ is a hyperedge in R if and only if the subgraph $G[r]$ induced by r is connected.

Proposition 17. *A coloring $c : V \rightarrow \mathbb{N}$ is a vertex ranking of G iff it is a unique-max coloring of H . Hence, $\chi_{um}(H) = \chi_{vr}(G)$.*

In particular, every vertex ranking of the path P_n is a unique-max coloring of the points with respect to intervals.

The following corollary characterizes the competitive ratio for the online hitting set problem for H in terms of the vertex ranking number of G . In fact, Propositions 5 and 6 imply the following corollary for special case of the path P_n .

Corollary 18. $\rho(H) = \chi_{vr}(G)$.

Proof. Follows from Coro. 8 and the Proposition 17. \square

Corollary 18 implies optimal competitive ratios of online hitting set algorithms for a wide class of graphs that admit (hereditary) small balanced separators. For example, consider the online hitting set problem for connected subgraphs of a given planar graph. Let G be a planar graph on n vertices. It was proved in [KMS95] that $\chi_{vr}(G) = O(\sqrt{n})$. Therefore, Coro. 18 implies that the competitive ratio of our algorithm for connected

Table 1: A list of several graph classes with small separators ($n = |V|$)

graph $G = (V, E)$	competitive ratio	previous result [AAA ⁺ 09]
path P_n	$\lfloor \log_2 n \rfloor + 1$	$O(\log^2 n)$
tree	$O(\log(\text{diameter}(G)))$	$O(n)$
tree-width d	$O(d \log n)$	$O(n)$
planar graph	$O(\sqrt{n})$	$O(n)$

subgraphs of planar graphs is $O(\sqrt{n})$. Corollary 18 also implies that this bound is optimal. Indeed, it was proved in [KMS95] that for the $l \times l$ grid graph $G_{l \times l}$ (with l^2 vertices), $\chi_{\text{vr}}(G_{l \times l}) \geq l$. Hence, for $G_{l \times l}$, any deterministic online hitting set algorithm must have a competitive ratio at least l . In Table 1 we list several important classes of such graphs.

We note that in the case of a star (i.e., a vertex v with $n - 1$ neighbors), the number of subsets of vertices that induce a connected graph is 2^{n-1} . Hence, the VC-dimension of the hypergraph is linear. However, the star has a vertex ranking that uses just two colors, hence, the competitive ratio of our algorithm in this case is 2. This is an improvement over the analysis of the algorithm of Alon et al. [AAA⁺09] which only proves a competitive ratio of $O(n)$. Thus, our algorithm is useful even in hypergraphs whose VC-dimension is unbounded.

6 Points and Half-Planes

In this section we consider a special instance of the online hitting set problem for a finite set of points in the plane and ranges induced by half-planes.

We prove the following results for hypergraphs in which the ground set X is a finite set of n points in \mathbb{R}^2 and the ranges are all subsets of X that can be cut off by a half-plane. Namely, a subset of points that lie above (respectively, below) a given line ℓ .

We note that the hypergraph of points and half-planes is not I -type. See Figure 1 for an example. Thus, Theorem 7 is not immediately applicable.

Theorem 19. *The competitive ratio of every online hitting set algorithm for points and half-planes is $\Omega(\log n)$.*

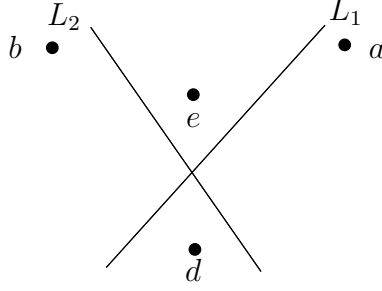


Figure 1: Two intersecting half-planes, the union of which is not induced by a half-plane. The half-planes $r_1 = \{a, d\}$ and $r_2 = \{b, d\}$ intersect. By convexity, every half-plane that contains a, b , and d must contain e .

Theorem 20. *There exists an online hitting set algorithm for points and half-planes that achieves a competitive ratio of $O(\log n)$.*

In the proofs we consider only ranges of points that are below a line; the case of points above a line is dealt with separately. This increases the competitive ratio by at most a factor of two.

Notation. Given a finite planar set of points X , let $V \subseteq X$ denote the subset of extreme points of X . That is, V consists of all points $p \in X$ such that there exists a half-plane h with $h \cap X = \{p\}$. Let $\{p_i\}_{i=1}^{|V|}$ denote an ordering of V in ascending x -coordinate order. Let $P = (V, E_P)$ denote the path graph over V where p_i is a neighbor of p_{i+1} for $i = 1, \dots, |V| - 1$. The intersection of every half-plane with V is a subpath of P . Namely, the intersection of a nonempty range r_i with V is a set of the form $\{p_j \mid j \in [a_i, b_i]\}$. We refer to such an intersection as a discrete interval (or simply an interval, if the context is clear). We often abuse this notation and refer to a point $p_i \in V$ simply by its index i . Thus, the interval of points in the intersection of r_i and V is denoted by $I_i \triangleq [a_i, b_i]$.

6.1 Proof of Theorem 19

We reduce the instance of intervals on a line (or equivalently, the path P_n and its induced connected subgraphs) to an instance of points and half-planes. Simply place the n points on the parabola $y = x^2$. Namely, point i is mapped to the point (i, i^2) . An interval $[i, j]$

of vertices is obtained by points below the line passing through the images of i and j . Hence, the problem of online hitting ranges induced by half-planes is not easier than the problem of online hitting intervals of P_n . The theorem follows from Proposition 6.

6.2 Proof of Theorem 20

Algorithm Description. The algorithm reduces the minimum hitting set problem for points and half-planes to a minimum hitting set of intervals in a path. The reduction is to the path graph P over the extreme points V of X . To apply Algorithm ALG_c (see Sec. 4.1), a vertex ranking c for P is computed, and each half-plane r_i is reduced to the interval I_i . A listing of Algorithm HS_p appears as Algorithm 1. Note that the algorithm HS_p uses only the subset $V \subset X$ of extreme points of X .

Algorithm 1 $\text{HS}_p(\{r_i\})$ - an online hitting set for points and half-planes

Require: $X \subset \mathbb{R}^2$ is a set of n points, and each r_i is an intersection of X with a half-plane.

- 1: $V \leftarrow$ the extreme points of X (i.e., lower envelope of the convex hull).
 - 2: $\{p_i\}_{i=1}^{|V|} \leftarrow$ ordering of V in ascending x -coordinate order.
 - 3: Let $P = (V, E_P)$ denote the path graph over V , where $E_P \triangleq \{(p_i, p_{i+1})\}_{i=1}^{|V|-1}$.
 - 4: $c \leftarrow$ a vertex ranking of P (with $\lfloor \log_2 |V| \rfloor + 1$ colors).
 - 5: Upon arrival of range r_i , reduce it to the interval $I_i = r_i \cap V$.
 - 6: Run ALG_c with the sequence of ranges $\{I_i\}_i$.
-

Analysis of the Competitive Ratio. The analysis follows the proof of Proposition 6. Recall that $\sigma(a)$ denotes the subsequence of σ consisting of ranges r_i that are unstabbed upon arrival and stabbed initially by a point colored a .

Lemma 21. *The ranges in $\sigma(a)$ are pairwise disjoint.*

Proof. Assume for the sake of contradiction that $r_i, r_j \in \sigma(a)$ and $z \in r_i \cap r_j$. Let $[a_i, b_i]$ denote the endpoints of the interval $I_i = r_i \cap V$, and define $[a_j, b_j]$ and I_j similarly. The proof of Proposition 6 proves that $I_i \cup I_j$ is not an interval. This implies that $z \notin V$ and that there is an extreme point $t \in V$ between I_i and I_j .

Without loss of generality, $b_i < t < a_j$. Let $(z)_x$ denote the x -coordinate of point z . Assume that $(z)_x \leq (t)_x$ (the other case is handled similarly). See Fig. 2 for an

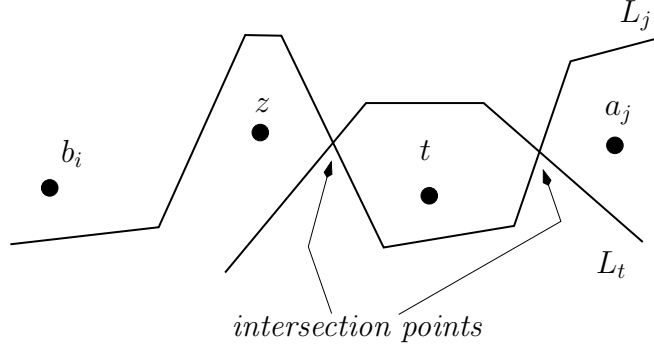


Figure 2: Proof of Lemma 21. The lines L_j and L_t are depicted as polylines only for the purpose of depicting their above/below relations with the points.

illustration. Let L_j denote a line that induces the range r_j , i.e., the set of points below L_j is r_j . Let L_t denote a line that separates t from $X \setminus \{t\}$, i.e., t is the only point below L_t . Then, L_t passes below z , above t , and below a_j . On the other hand, L_j passes above z , below t , and above a_j . Since $(z)_x \leq (t)_x < (a_j)_x$, it follows that the lines L_t and L_j intersect twice, a contradiction, and the lemma follows. \square

Lemma 21 implies that $|\text{OPT}(\sigma)| \geq \max_a |\sigma(a)|$. On the other hand $|\text{HS}_p(\sigma)| = \sum_a |\sigma(a)| \leq (1 + \log n) \cdot \max_a |\sigma(a)|$, and Theorem 20 follows.

7 Points and Unit Discs

In this section we consider a special instance of the online hitting set problem in which the ground set X is a finite set of n points in \mathbb{R}^2 . The ranges are subsets of points that are contained in a unit disc. Formally, a unit disc d centered at o is the set $d \triangleq \{x \in \mathbb{R}^2 : \|x - o\|_2 \leq 1\}$. The range $r(d)$ induced by a disc d is the set $r(d) \triangleq \{x \in X : x \in d\}$. The circle ∂d is defined by $\partial d \triangleq \{x \in \mathbb{R}^2 : \|x - o\|_2 = 1\}$.

As in the case of points and half-planes, the hypergraph of points and unit discs is not I -type. To see this, assume that the distances between the four points in Fig. 1 are small. In this case, the lines L_1 and L_2 can be replaced by unit discs that induce the same ranges.

Theorem 22. *The competitive ratio of every online hitting set algorithm for points and unit discs is $\Omega(\log n)$.*

Proof. Reduce an instance of intervals on a line to points and half-planes as follows. Position n points on a line such that the distance between the first and last point is less than one. For each interval, there exists a unit disc that intersects the points in exactly the same points as the interval. Thus, the lower bound for points and intervals (Prop. 6) holds also for unit discs. \square

Theorem 23. *There exists an online hitting set algorithm for points and discs that achieves a competitive ratio of $O(\log n)$.*

7.1 Proof of Theorem 23

Partitioning. We follow Chen et al. [CKS09] with the following partitioning of the plane (see Fig. 3). Partition the plane into square tiles with side-lengths $1/2$. Consider a square s in this tiling. Let S denote a square concentric with s whose side length is $5/2$. Partition S into four quadrants, each a square with side length $5/4$. Let S^i denote a quadrant of S and let o^i denote its center, for $i \in \{1, 2, 3, 4\}$. Let D_s denote the set of unit discs d such that $d \cap s \neq \emptyset$.

Proposition 24. *If $d \in D_s$, then $d \cap \{o^1, \dots, o^4\} \neq \emptyset$.*

For $d \in D_s$, let $\tau(s, d) \triangleq \min\{i : o^i \in d\}$. For $\tau \in \{1, \dots, 4\}$, let $D_{s,\tau}$ denote the set $\{d \in D_s \mid \tau(s, d) = \tau, d \cap s \cap X \neq \emptyset\}$. The following lemma shows that circles bounding the discs in $D_{s,\tau}$ behave like pseudo-lines when restricted to a subregion of S .

Lemma 25 ([CKS09]). *Let $K^{s,\tau}$ denote the convex cone with apex o^τ spanned by s . Then, for any pair of discs $d, d' \in D_{s,\tau}$, the circles ∂d and $\partial d'$ intersect at most once in $K^{s,\tau}$.*

Extreme points. For every square tile s and every $\tau \in \{1, \dots, 4\}$, we define a set $V_{s,\tau}$ of *extreme points* as follows.

$$V_{s,\tau} \triangleq \{x \in X \mid \exists d \in D_{s,\tau} : d \cap s \cap X = \{x\}\}.$$

Note that if $d \in D_{s,\tau}$ and $d \cap s \cap X \neq \emptyset$, then $d \cap V_{s,\tau} \neq \emptyset$.

Let $\theta_{s,\tau} : V_{s,\tau} \rightarrow [0, 2\pi]$ denote an *angle function*, where $\theta_{s,\tau}(x)$ equals the slope of the line $o^\tau x$. Let $\{p_i\}_{i=1}^{|V_{s,\tau}|}$ denote an ordering of $V_{s,\tau}$ in increasing $\theta_{s,\tau}$ order. For a disc

update $C_i \leftarrow C_{i-1}$. Otherwise, a vertex $v_{i,s}$ is selected from each square tile s such that $r_i \cap s \neq \emptyset$. These vertices are added to C_{i-1} to obtain C_i .

Lemma 25 provides an interpretation of Algorithm HS_d as a reduction to the case of hitting subsets of points below a pseudo-line (i.e., pseudo half-planes). Each square tile s and type $\tau \in \{1, \dots, 4\}$ defines an instance of points and pseudo half-planes with respect to the set $X_s \triangleq X \cap s$ of points and the subsets $d \cap X_s$ for discs $d \in D_{s,\tau}$. The algorithm maintains a different invocation of HS_p for each square s and type τ . Upon arrival of an unstabbed disc d , the algorithm inputs the range $d \cap s \cap X$ to each instance of HS_p corresponding to a square s and a type τ such that $d \in D_{s,\tau}$.

Algorithm 2 $\text{HS}_d(X)$ - an online hitting set for unit discs.

Require: $X \subset \mathbb{R}^2$ is a set of n points. A tiling by $1/2 \times 1/2$ squares. Four types of extreme points $V_{s,\tau}$ per tile. A vertex ranking $c^{s,\tau}$ of $V_{s,\tau}$ with respect to the “angular” order.

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1:  $C_0 \leftarrow \emptyset$ 
2: for  $i = 1$  to  $\infty$  do {arrival of a range  $r_i = r(d_i)$ }
3:   if  $r_i$  not stabbed by  $C_{i-1}$  then
4:     for all square tiles  $s$  such that  $r_i \cap s \neq \emptyset$  do
5:        $\tau \leftarrow \tau(s, d_i)$  {find the type of  $d_i$  wrt  $s$ }
6:        $v_{s,i} \leftarrow v_{\max}^{s,\tau}(r_i \cap V_{s,\tau})$  {find the vertex with the max color}
7:        $C_i \leftarrow C_{i-1} \cup \{v_{s,i}\}$ 
8:     end for
9:   else
10:     $C_i \leftarrow C_{i-1}$ 
11:   end if
12: end for

```

7.1.2 Analysis of The Competitive Ratio

Let $\sigma = \{r_i\}_i$ denote the input sequence. Let $\sigma^A \subseteq \sigma$ denote the subsequence of ranges r_i such that r_i is unstabbed upon arrival (i.e., r_i is not stabbed by C_{i-1}).

Proposition 27. $|\text{HS}_d(\sigma)| \leq 16 \cdot |\sigma^A|$.

Proof. Each disc intersects at most 16 square tiles. Upon arrival of an unstabbed disc, at most one point is added to the hitting set, for each intersected square. \square

The following lemma shows that, if two discs contain a common point $x \in s$, are of the same type τ , and are unstabbed upon arrival, then they are stabbed by extreme points in $V_{s,\tau}$ of different colors.

Lemma 28. *If $x \in X \cap s$, $r_i, r_j \in \sigma^A \cap D_{s,\tau}$ and $x \in r_i \cap r_j$, then $c^{s,\tau}(v_{s,i}) \neq c^{s,\tau}(v_{s,j})$.*

Proof. To shorten notation let $c = c^{s,\tau}$, $V = V_{s,\tau}$, and $\theta = \theta^{s,\tau}$. Assume for the sake of contradiction that $c(v_{s,i}) = c(v_{s,j})$. By Prop. 26, $r_i \cap V$ is an interval, which we denote by $I_i = [a_i, b_i]$. Similarly, $I_j = [a_j, b_j]$ is the interval for $r_j \cap V$. Since $c^{s,\tau}$ is a unique-max coloring of the intervals in $V_{s,\tau}$, $I_i \cup I_j$ is not an interval, so there must be an extreme point in between the intervals. Denote this in between point by t . Without loss of generality, $\theta(b_i) < \theta(t) < \theta(a_j)$. Assume that $\theta(z) \leq \theta(t)$. Consider a disc $d_j \in D_{s,\tau}$ such that $r_j = r(d_j)$. Consider a disc $d_t \in D_{s,\tau}$ such that $d_t \cap X_s = \{t\}$. We claim that the circles ∂d_j and ∂d_t intersect twice in the cone $K^{s,\tau}$, contradicting Lemma 25. Indeed, ∂d_t passes “below” x , “above” t , and “below” a_j . On the other hand, ∂d_j passes above x , below t , and above a_j . The case $\theta(z) > \theta(t)$ is proved similarly by considering the discs d_t and d_i . \square

Let $\sigma(x)$ denote the subsequence of ranges r_i such that $x \in r_i$. The following lemma proves that the algorithm stabs a sequence of discs that share a common point by $O(\log n)$ points.

Lemma 29. *For every $x \in X$, $|\text{HS}_d(\sigma(x))| \leq 64 \cdot \lfloor \log_2(2n) \rfloor$.*

Proof. Fix a point $x \in X$, and let s denote the tile such that $x \in s$. Let $\sigma^A(x)$ denote the sequence of ranges in $\sigma(x)$ that were unstabbed upon arrival in an execution of $\text{ALG}(\sigma(x))$. By Prop. 27, $|\text{HS}_d(\sigma(x))| \leq 16 \cdot |\sigma^A(x)|$.

The disc d_i of each range $r_i \in \sigma^A(x)$ belongs to one of four types $D_{s,\tau}$, for $1 \leq \tau \leq 4$. By Lemma 28, the ranges in $\sigma^A(x) \cap D_{s,\tau}$ are stabbed by extreme points in $V_{s,\tau}$, the colors of which are distinct. Each vertex ranking $c^{s,\tau}$ uses at most $\lfloor \log_2(2n) \rfloor$ colors. Thus, $|\sigma^A(x)| \leq \sum_{\tau=1}^4 |\sigma^A(x) \cap D_{s,\tau}| \leq 4 \cdot \lfloor \log_2(2n) \rfloor$, and the lemma follows. \square

Proof of Theorem 23. Consider an execution of $\text{HS}_d(\sigma)$ and independent executions of $\text{HS}_d(\sigma(x))$, for every $x \in \text{OPT}(\sigma)$. Every time $\text{HS}_d(\sigma)$ is input an unstabbed range r_i , at least one of the executions of $\text{HS}_d(\sigma(x))$ is also input r_i , and r_i is also unstabbed upon arrival. This implies that $|\text{HS}_d(\sigma)| \leq \sum_{x \in \text{OPT}(\sigma)} |\text{HS}_d(\sigma(x))|$.

By Lemma 29, $|\text{HS}_d(\sigma(x))| = O(\log n)$. This implies that $|\text{HS}_d(\sigma)| = O(\log n) \cdot |\text{OPT}(\sigma)|$, and the theorem follows. \square

8 Discussion

We would like to suggest two open problems.

1. Design an online hitting set algorithm for points and arbitrary discs, the competitive ratio of which is $o(\log^2 n)$ or prove a lower bound of $\Omega(\log^2 n)$.
2. Design an online hitting set algorithm with a logarithmic competitive ratio for any hypergraph with bounded VC-dimension or obtain a lower bound as above. Alon et al. obtain an $O(\log^2 n)$ competitive ratio, and the best known lower bound is $\Omega(\log n)$.

Acknowledgments

We would like to thank Nabil Mustafa and Saurabh Ray for discussions on the open problems suggested in the last section.

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